Value Based - Part 1

Deep Reinforcement Learning

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Courtesy: Some slides are adopted from CS 285 Berkeley, and CS 234 Stanford, and Pieter Abbeel's compact series on RL.

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Value Function

- $V^*(s)$ = expected utility starting in s, and acting optimally in all subsequent actions.

$$V^*(s) = \max_{a_i' s} \mathbb{E}\left(\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t, s_{t+1}) \middle| s_0 = s\right)$$

Value Iteration

- $V_0^*(s)$ = optimal value for state s when H=0
  - ☑ $V_0^*(s) = 0 \forall s$
- $V_1^*(s)$ = optimal value for state s when H=1
  - ☑ $V_1^*(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_0^*(s'))$
- $V_2^*(s)$ = optimal value for state s when H=2
  - ☑ $V_2^*(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_1^*(s'))$
- $V_k^*(s)$ = optimal value for state s when H = k
  - ☑ $V_k^*(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_{k-1}^*(s'))$

Value Iteration

Algorithm:

Start with $V_0^*(s) = 0$ **for all s.**

For k = 1, … , H:

For all states s in S:

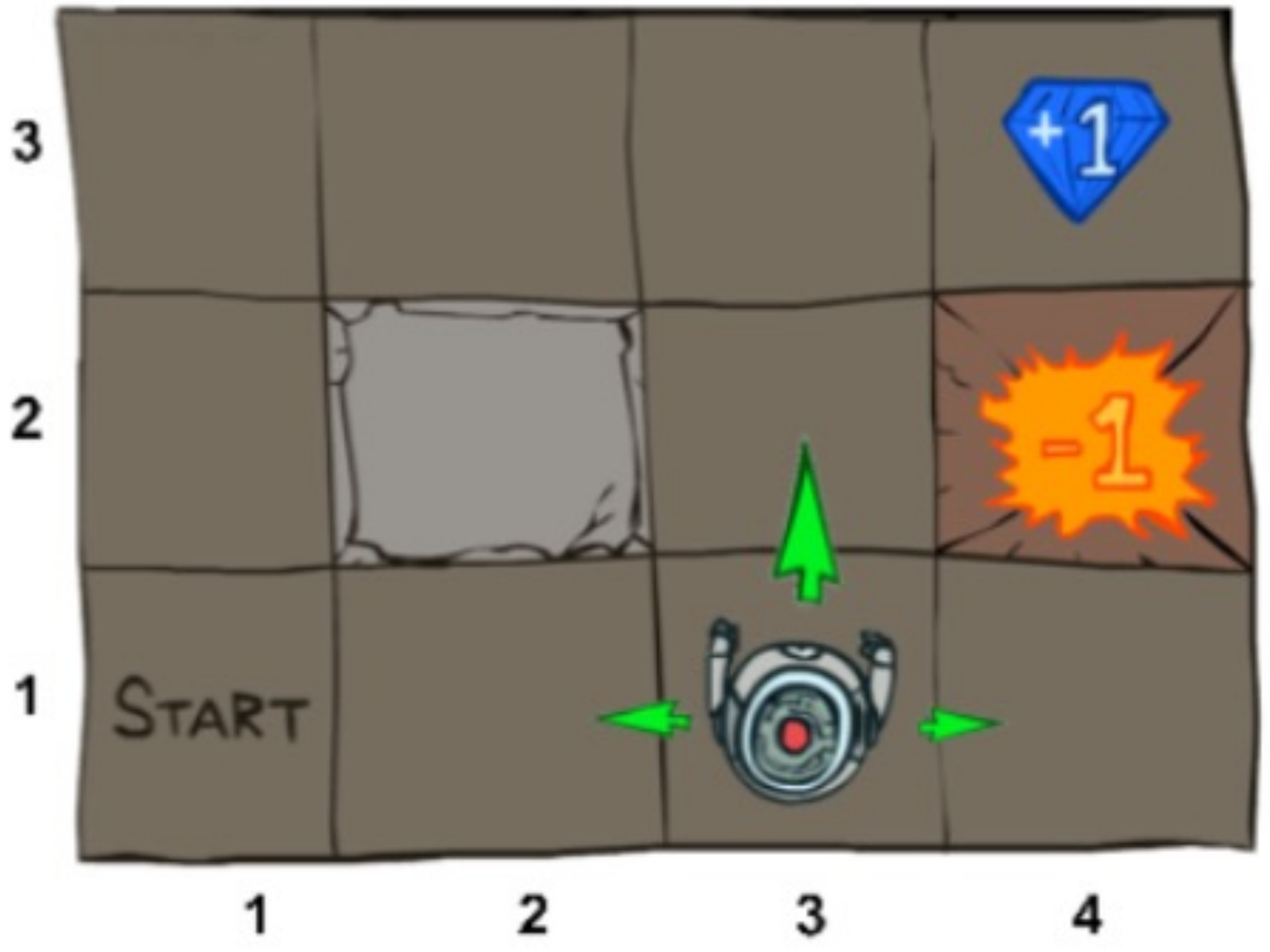
$$V_k^*(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_{k-1}^*(s'))$$

$$\pi_k^*(s) \leftarrow \mathrm{arg}\max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_{k-1}^*(s'))$$

☑ This is called a value update or Bellman update/back-up

Value Iteration

$$V_0(s) \leftarrow 0$$



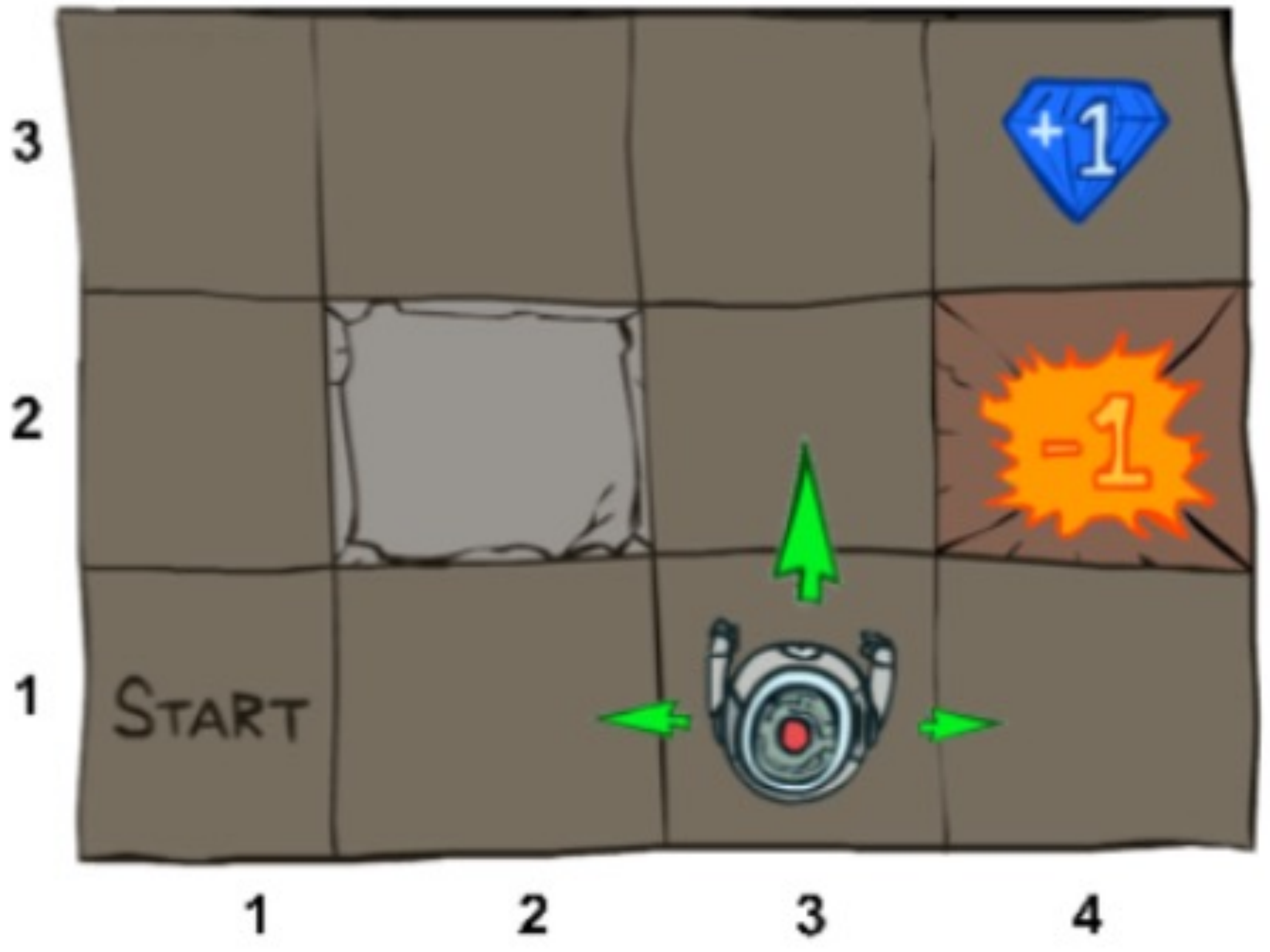
Noise = 0.2
Discount = 0.9

k = 0

Value Iteration

$$V_1(s) = \max_a \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma V_0(s')\right)$$

k = 0

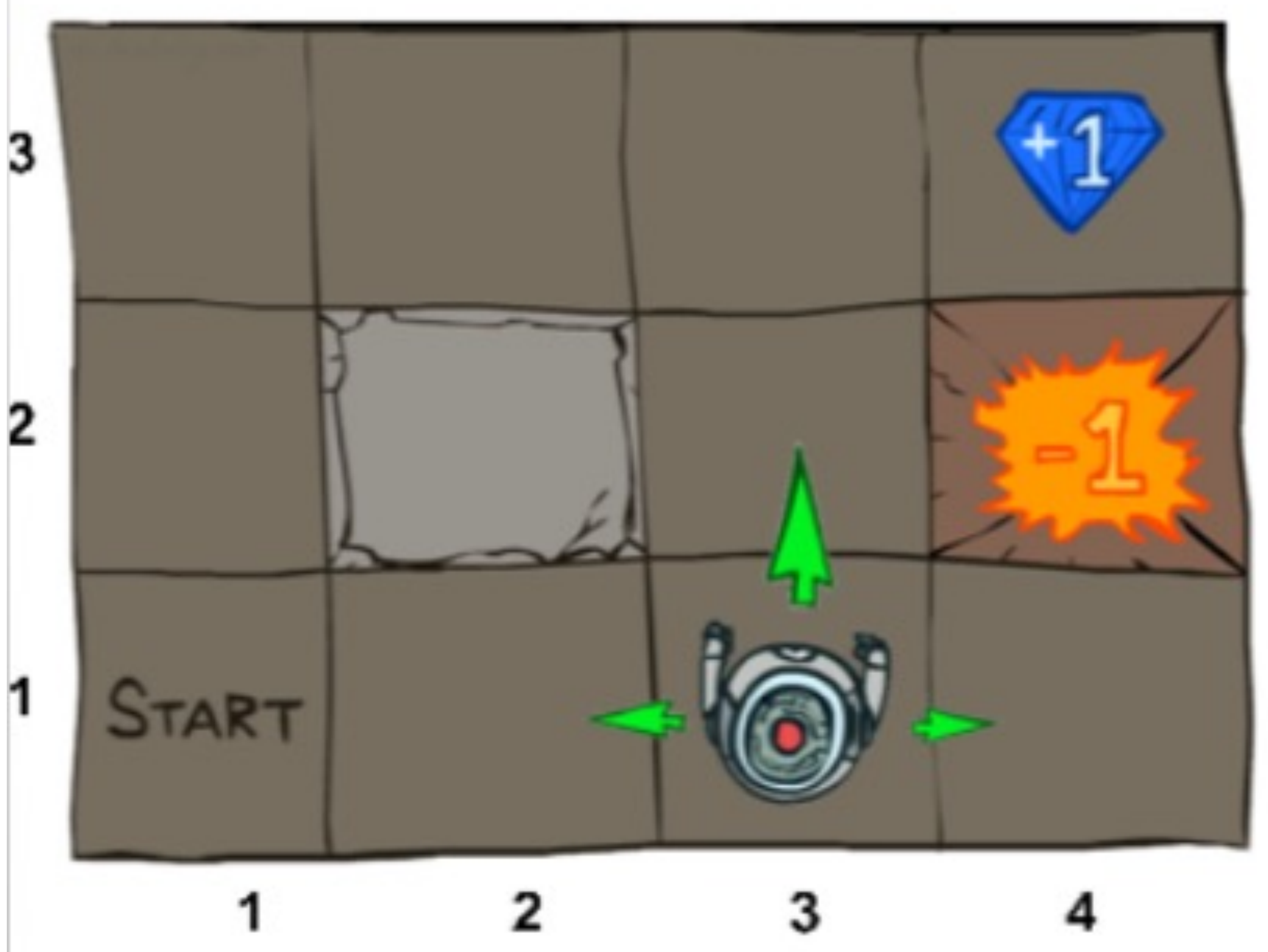


Noise = 0.2

Discount = 0.9

Value Iteration

$$V_2(s) = \max_a \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma V_1(s')\right)$$



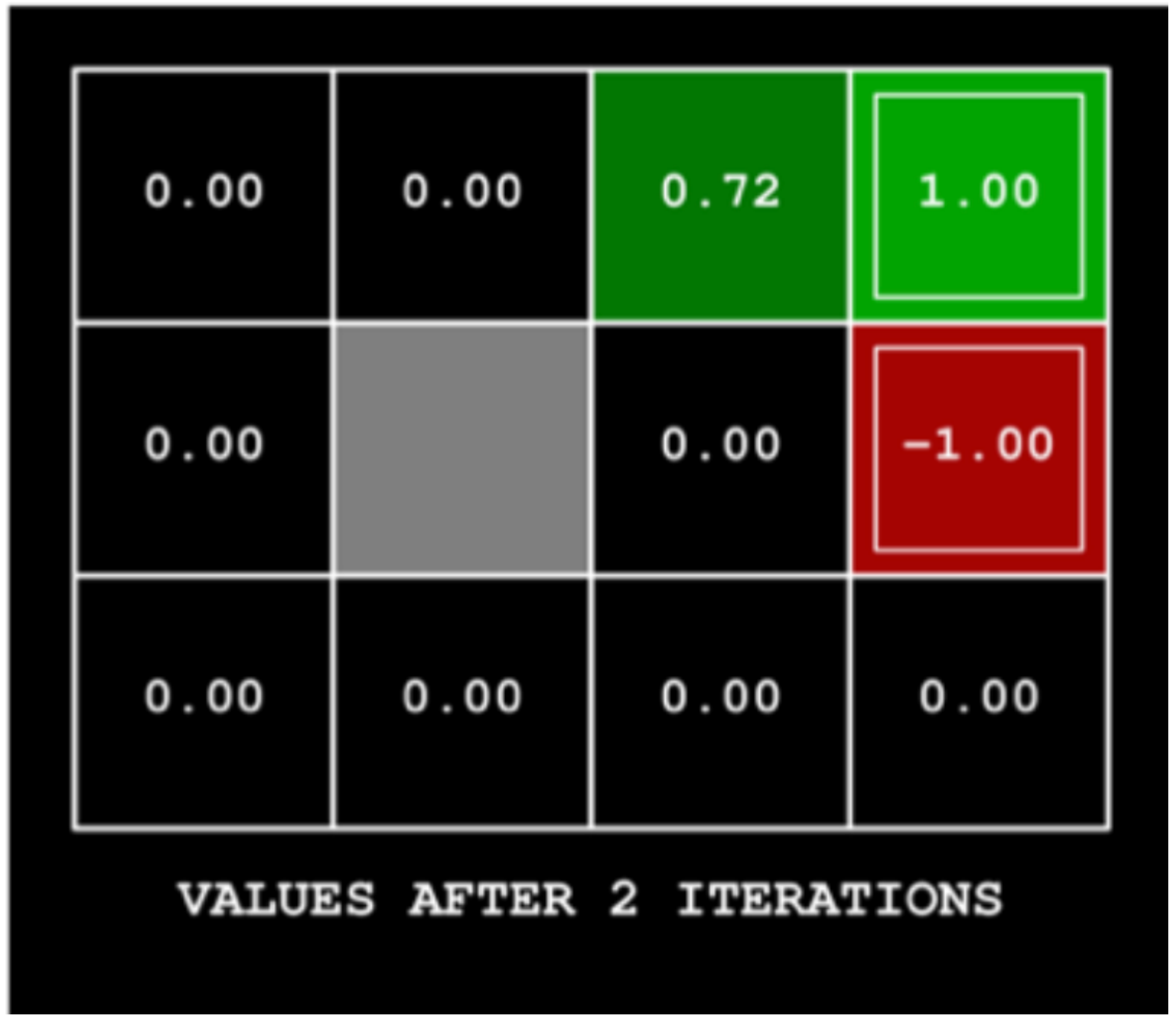
Noise = 0.2
Discount = 0.9

k = 1

Value Iteration

$$V_2(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_1(s'))$$

k = 2

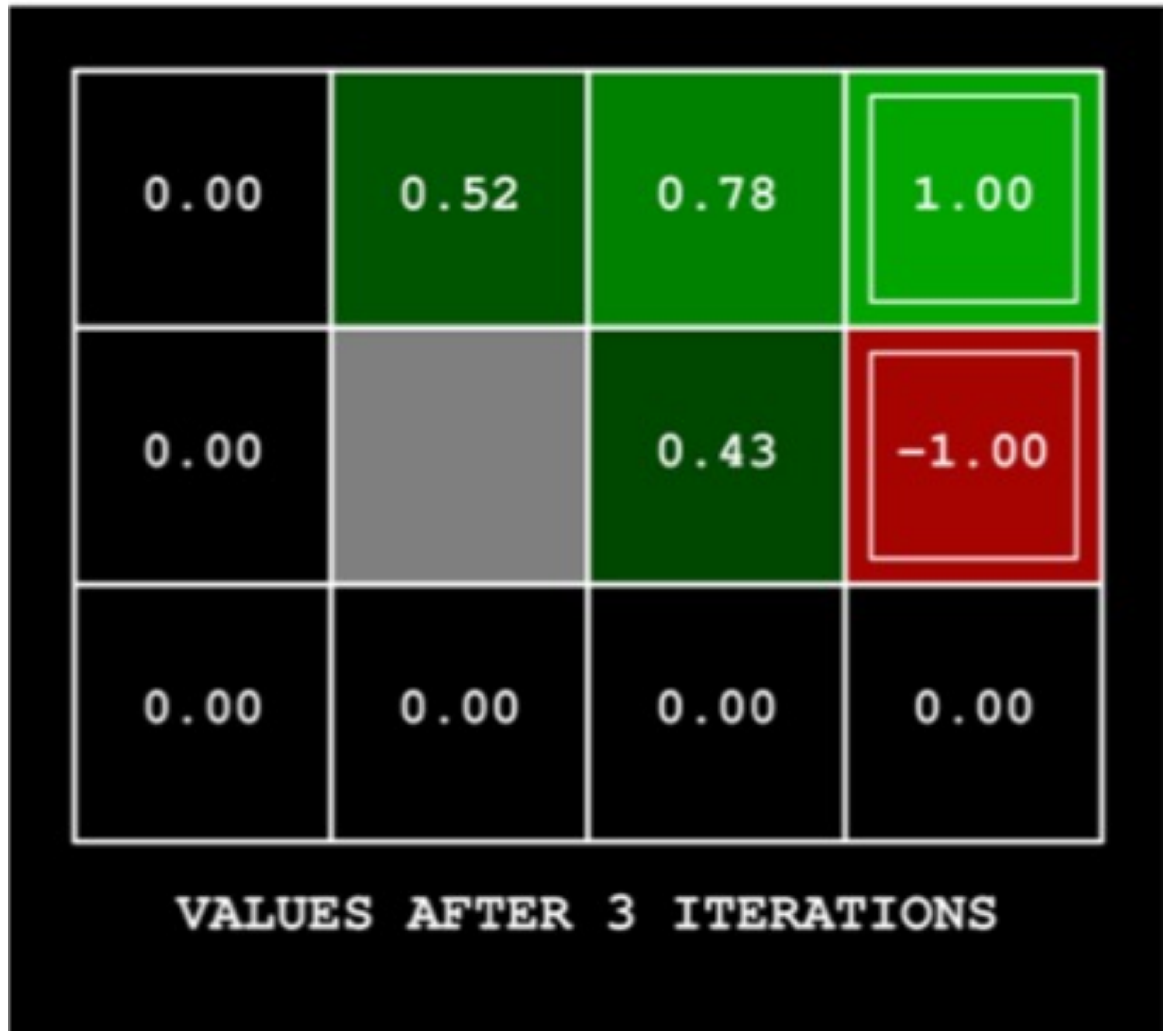


Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_{\mathrm{k}}(s'))$$

k = 3



Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) = \max_{a} \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_{k}(s'))$$

k = 4

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

k = 5

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_{\mathrm{k}}(s'))$$

k = 6

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

k = 7

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) = \max_{a} \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

k = 8

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) = \max_{a} \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_{k}(s'))$$

k = 9

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) = \max_{a} \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_{\mathrm{k}}(s'))$$

k = 10

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) = \max_{a} \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_{\mathrm{k}}(s'))$$

k = 11

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

k = 12

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) = \max_{a} \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_{\mathrm{k}}(s'))$$

k = 100

Noise = 0.2
Discount = 0.9

Q-Values

- $Q^*(s,a)$ = expected utility starting in s, taking action a, and (thereafter) acting optimally

$$V^*(s) = \max_{a'} Q^*(s,a')$$

- Bellman Equation:

$$Q^*(s,a) = \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma \max_{a'} Q^*(s',a'))$$

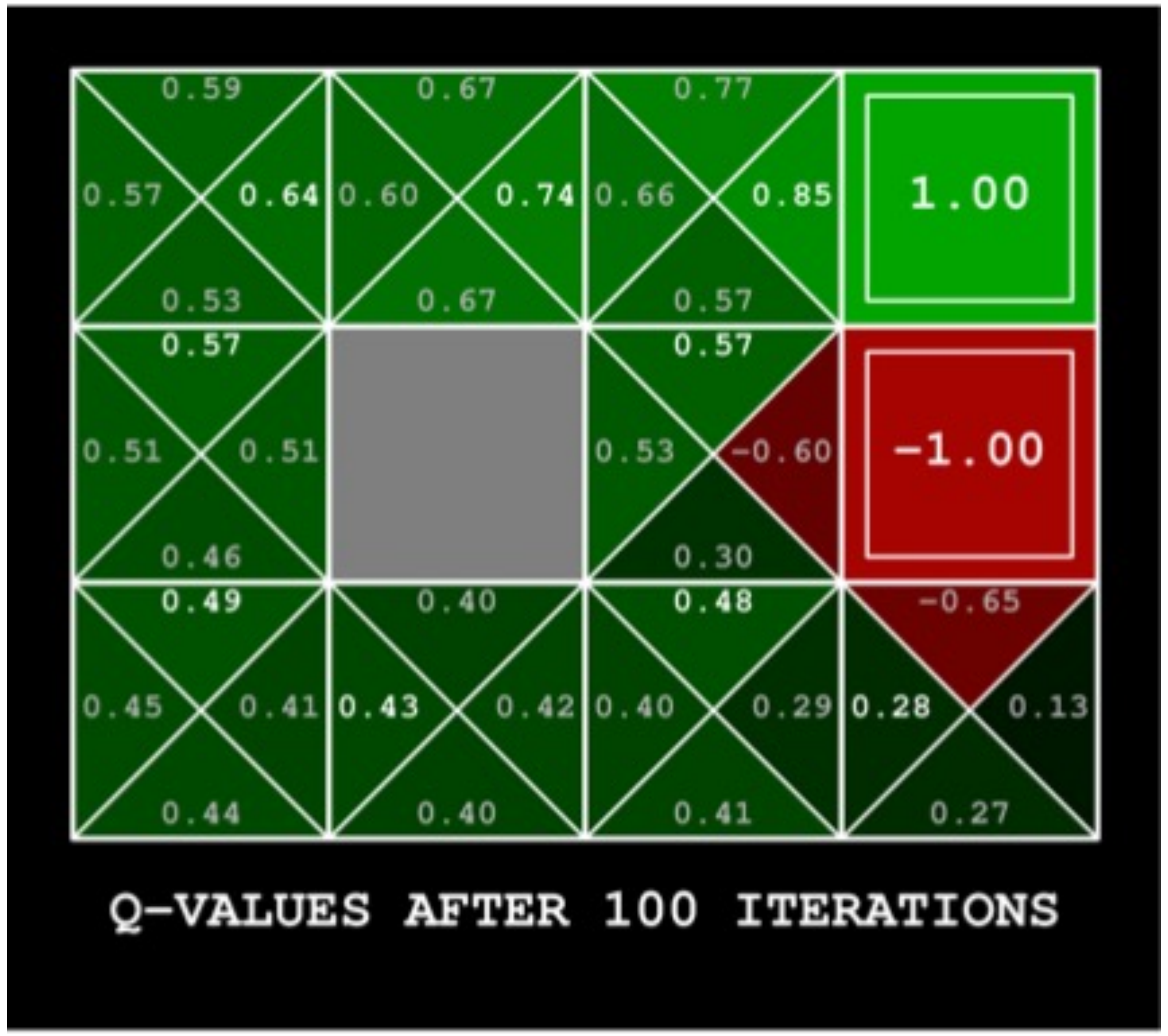
- Q-value Iteration:

$$Q^*_{k+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma \max_{a'} Q^*_k(s',a'))$$

Q-Value Iteration

$$Q_{k+1}^{*}(s,a) \leftarrow \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma \max_{a'} Q_k^{*}(s',a')\right)$$

k = 100



Noise = 0.2
Discount = 0.9

Policy Evaluation

- Recall value iteration:
  - ☑ $V_k^*(s) \leftarrow max_a \sum_{s'} P(s'|s,a)\big(R(s,a,s') + \gamma V_{k-1}^*(s')\big)$
- Policy evaluation for a given $\pi(s)$:
  - ☑ $V_k^\pi(s) \leftarrow \sum_{s'} P(s'|s,\pi(s))(R(s,\pi(s),s') + \gamma V_{k-1}^\pi(s'))$
- At convergence:
  - ☑ $\forall s V^\pi(s) \leftarrow \sum_{s'} P(s'|s,\pi(s))(R(s,\pi(s),s') + \gamma V^\pi(s))$

Policy Iteration

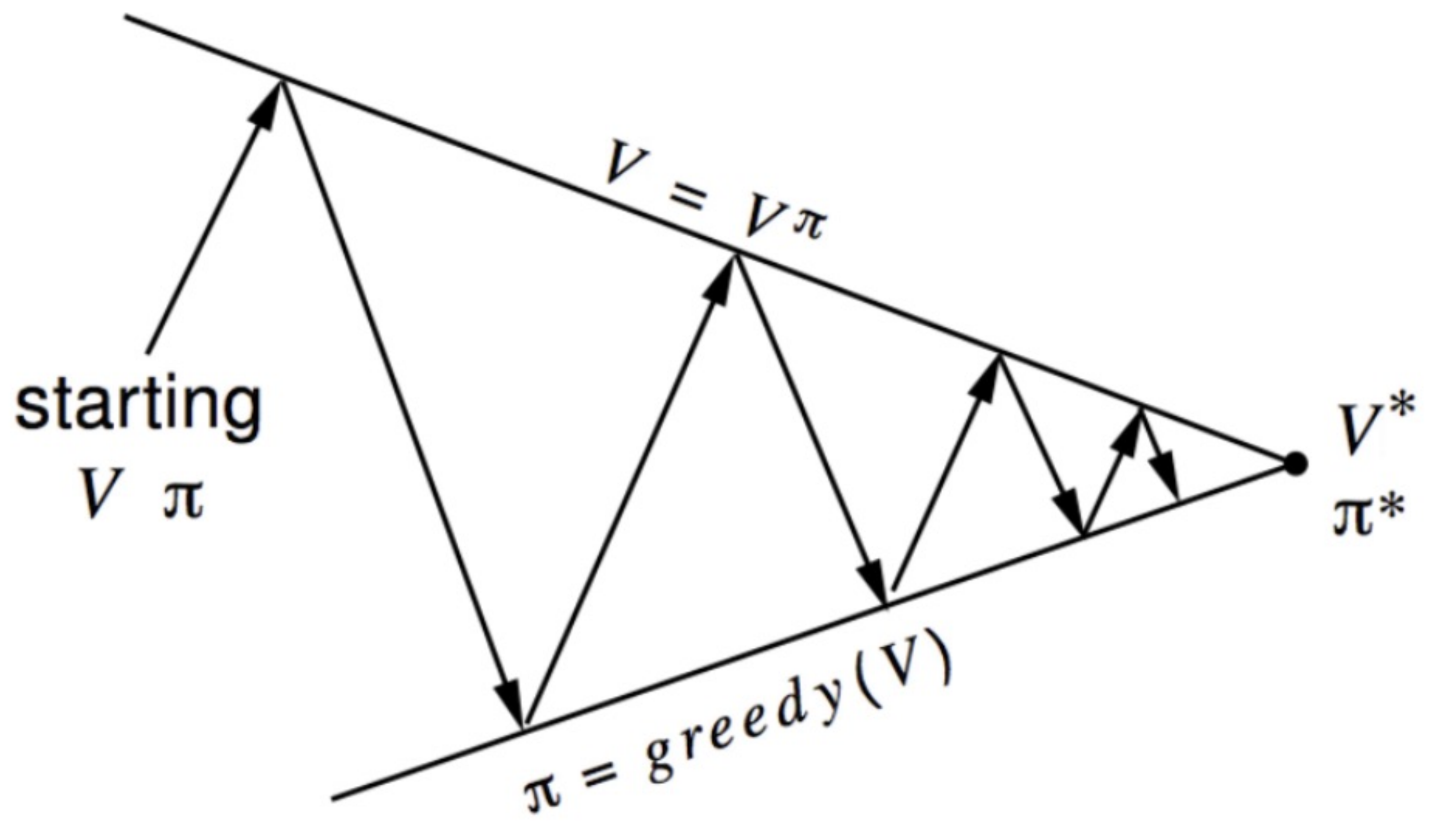
- One iteration of policy iteration
  - Policy evaluation for current policy π_k. Iterate until convergence

$$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} P(s'|s, \pi_k(s))[R(s, \pi_k(s), s') + \gamma V_i^{\pi_k}(s')]$$

  - Policy improvement: find the best action according to one-step look-ahead

$$\pi_{k+1}(s) \leftarrow \mathrm{arg}\max_a \sum_{s'} P(s'|s, a)[R(s, a, s') + \gamma V^{\pi_k}(s')]$$

- Repeat until policy converges
  - At convergence: optimal policy; and converges faster than value iteration under some conditions



One-step look ahead improves the policy

- Consider an alternative policy $\pi^{(1)}_{(k+1)}(t,s)$ that takes the prescribed actions in $\pi_{k+1}(s)$ only at time t = 0; and stays the same as $\pi_k(s)$ in later times.
- The value function V(s) for this new time-dependent policy is larger than or equal to V(s) for the original policy $\pi_k(s)$ for all s. Why?
- Now let $\pi^{(2)}_{(k+1)}(t,s)$, which takes the prescribed action in $\pi_{k+1}(s)$ only at times t = 0 and t = 1, and stays the same as $\pi_k(s)$ in later times.
- Similarly, V(s) gets improved for $\pi^{(2)}_{(k+1)}(t,s)$ compared to $\pi^{(1)}_{(k+1)}(t,s)$ for all s.
- Repeating this argument $\pi^{(\infty)}_{(k+1)}(t,s)$ becomes the same as $\pi_{k+1}(s)$.

An Example

- Let this be the initial policy π_0, show how policy improvement, makes this policy better.

+1	→	+1
-1	↑	-1
-1	→	-1
-1	↑	-1
-1	←	-1
-1	↑	-1
-1	→	-1

Planning vs. Learning

- Assumed to have access to the dynamics $P(s'|s,a)$.
- We don't have access to this in the real world.
- We need to estimate (or learn) the value functions.

$$Q^*(s,a) = \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma \max_{a'} Q^*(s',a'))$$

Monte-Carlo Prediction

Monte Carlo Methods - Introduction

- Experience samples to learn without a model

- MC methods require only experience— sample sequences of states, actions, and rewards from actual or simulated interaction with an environment.

- We can learn with samples: episodes!

We don't have access to $P(s'|s,a)$ → Model Free Learning!

Monte-Carlo prediction

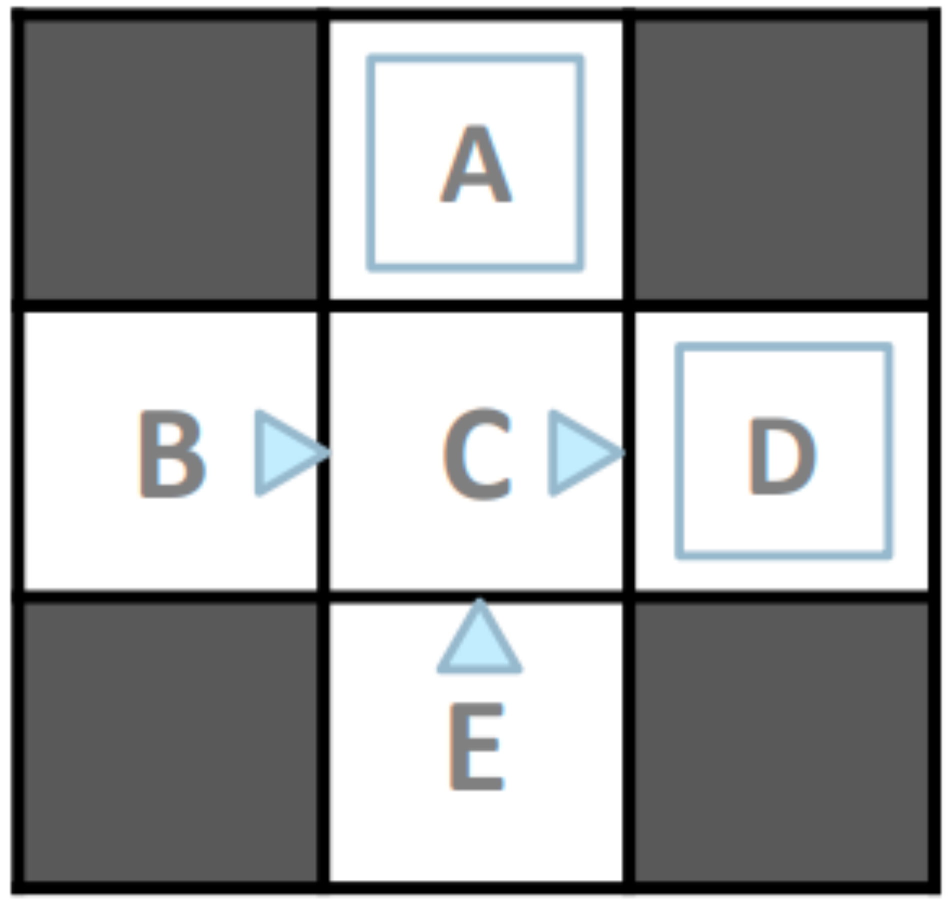
- Suppose we wish to estimate $V_\pi(s)$, the value of a state s under policy π.
- The first-visit mc method estimates $V_\pi(s)$ as the average of the returns following first visits to s.

First-visit MC prediction, for estimating $V \approx v_\pi$

```
Input: a policy π to be evaluated
Initialize:
    V(s) ∈ ℝ, arbitrarily, for all s ∈ 𝒮
    Returns(s) ← an empty list, for all s ∈ 𝒮
Loop forever (for each episode):
    Generate an episode following π: S_0, A_0, R_1, S_1, A_1, R_2, ..., S_{T-1}, A_{T-1}, R_T
     G ← 0
    Loop for each step of episode, t = T-1, T-2, ..., 0:
        G ← γG + R_{t+1}
        Unless S_t appears in S_0, S_1, ..., S_{t-1}:
            Append G to Returns(S_t)
             V(S_t) ← average(Returns(S_t))
```

Episodes: another example

Input Policy π



Assume: $\gamma = 1$

Observed Episodes (Training)

Episode 1

B, east, C, -1
C, east, D, -1
D, exit, x, +10

Episode 2

B, east, C, -1
C, east, D, -1
D, exit, x, +10

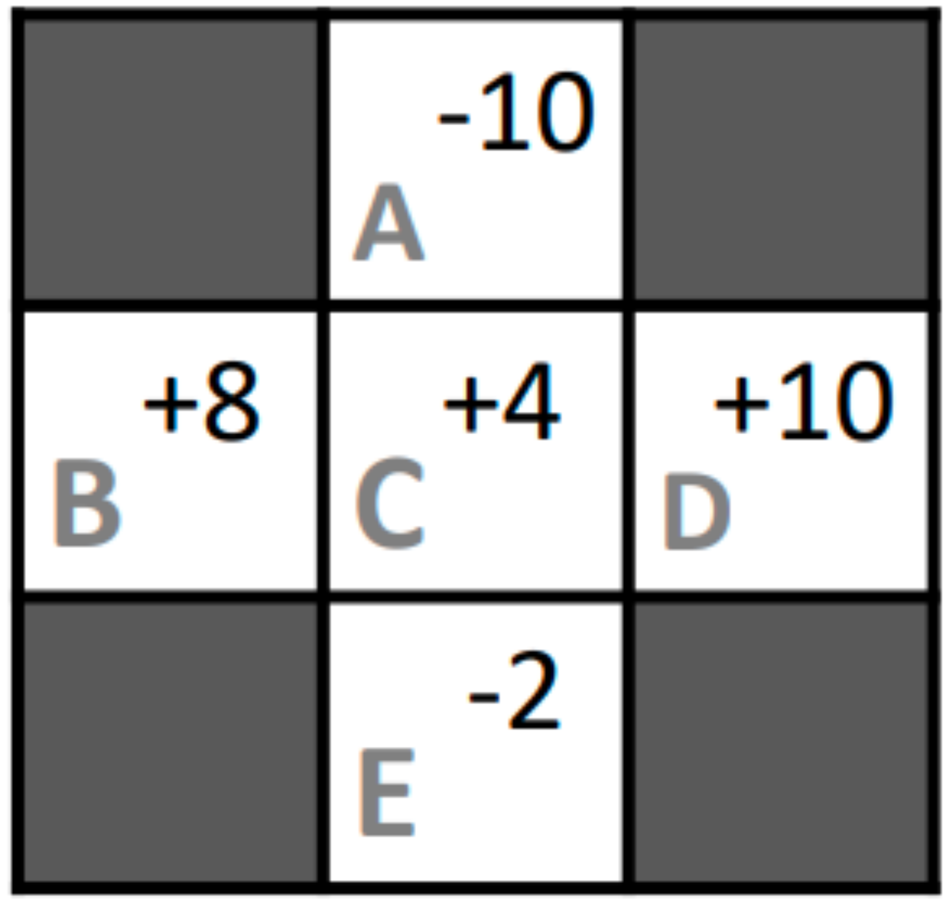
Episode 3

E, north, C, -1
C, east, D, -1
D, exit, x, +10

Episode 4

E, north, C, -1
C, east, A, -1
A, exit, x, -10

Output Values



Every Visit Monte-Carlo Policy

Every Visit Monte-Carlo Policy

Initialize $N(s) = 0, G(s) = 0 \forall s \in S$

Loop

Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$

Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ **as return from time step** t **onwards in i'th episode.**

For each time step t **until** T_i **(the end of the episode** i**):**

state s **is the state visited at time step** t **in episodes** i

Increment counter of total visits: $N(s) = N(s) + 1$

Increment total return: $G(s) = G(s) + G_{i,t}$

Update estimate $V^{\pi}(s) = G(s)/N(s)$

Incremental Monte-Carlo Policy

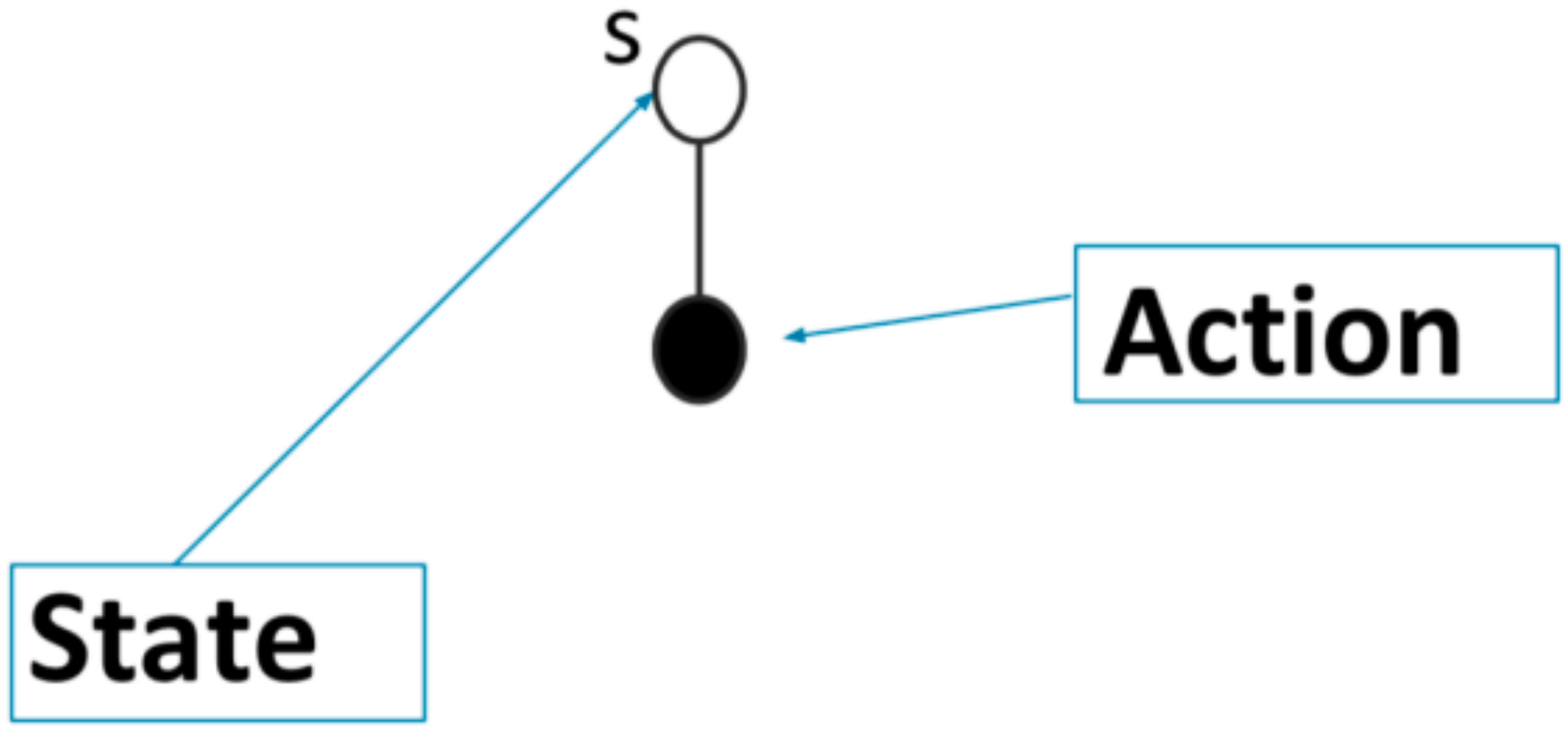
Incremental Monte-Carlo Policy

After each episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots$

- **Define** $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots$ **as return from time step** t **onwards in i'th episode**
- **For state** s **visited at time step** t **in episode** i
  - **Increment counter of total visits:** $N(s) = N(s) + 1$
  - **Update estimate**

$$V^{\pi}(s) = V^{\pi}(s)\frac{N(s)-1}{N(s)} + \frac{G_{i,t}}{N(s)} = V^{\pi}(s) + \frac{1}{N(s)}\left(G_{i,t} - V^{\pi}(s)\right)$$

Policy Evaluation Diagram



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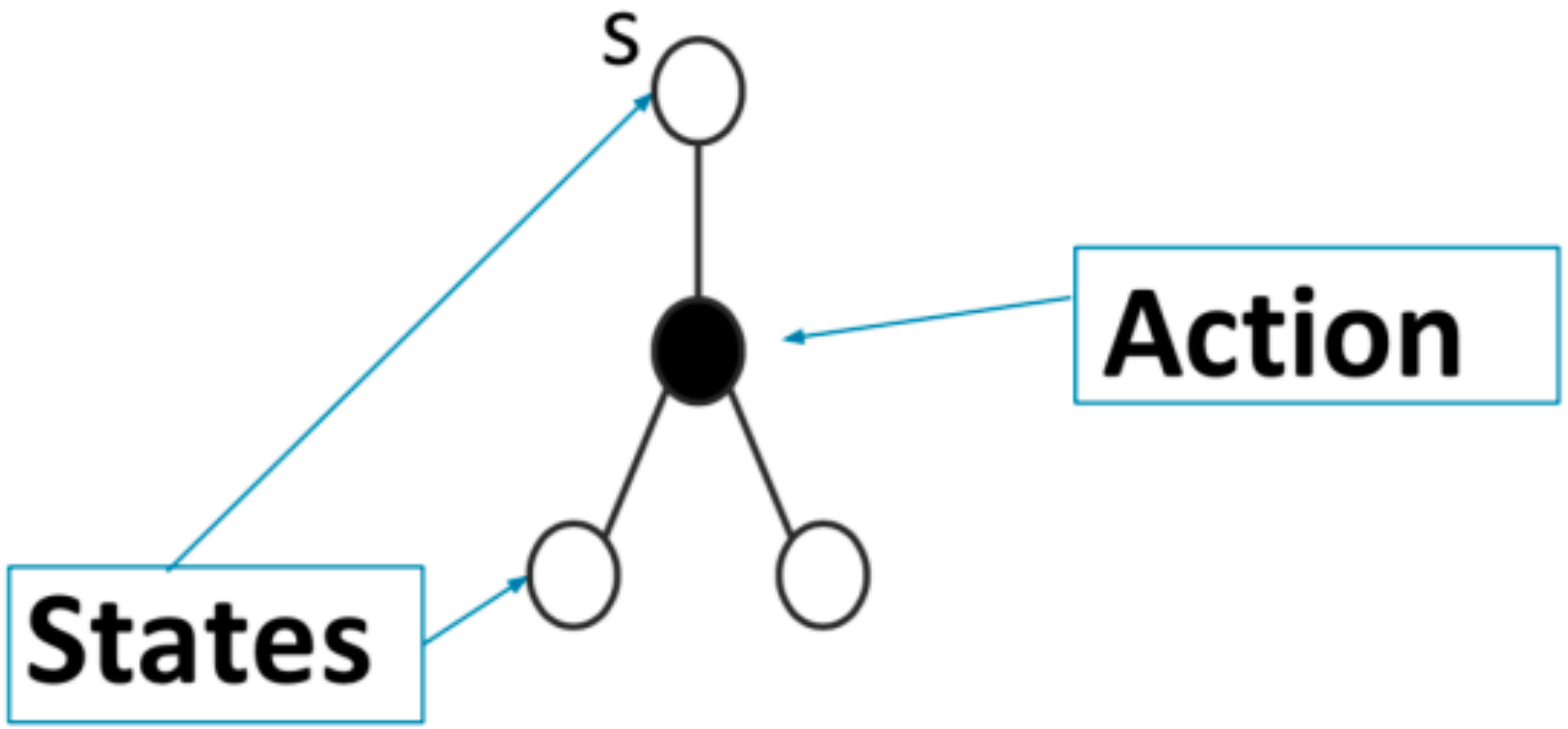
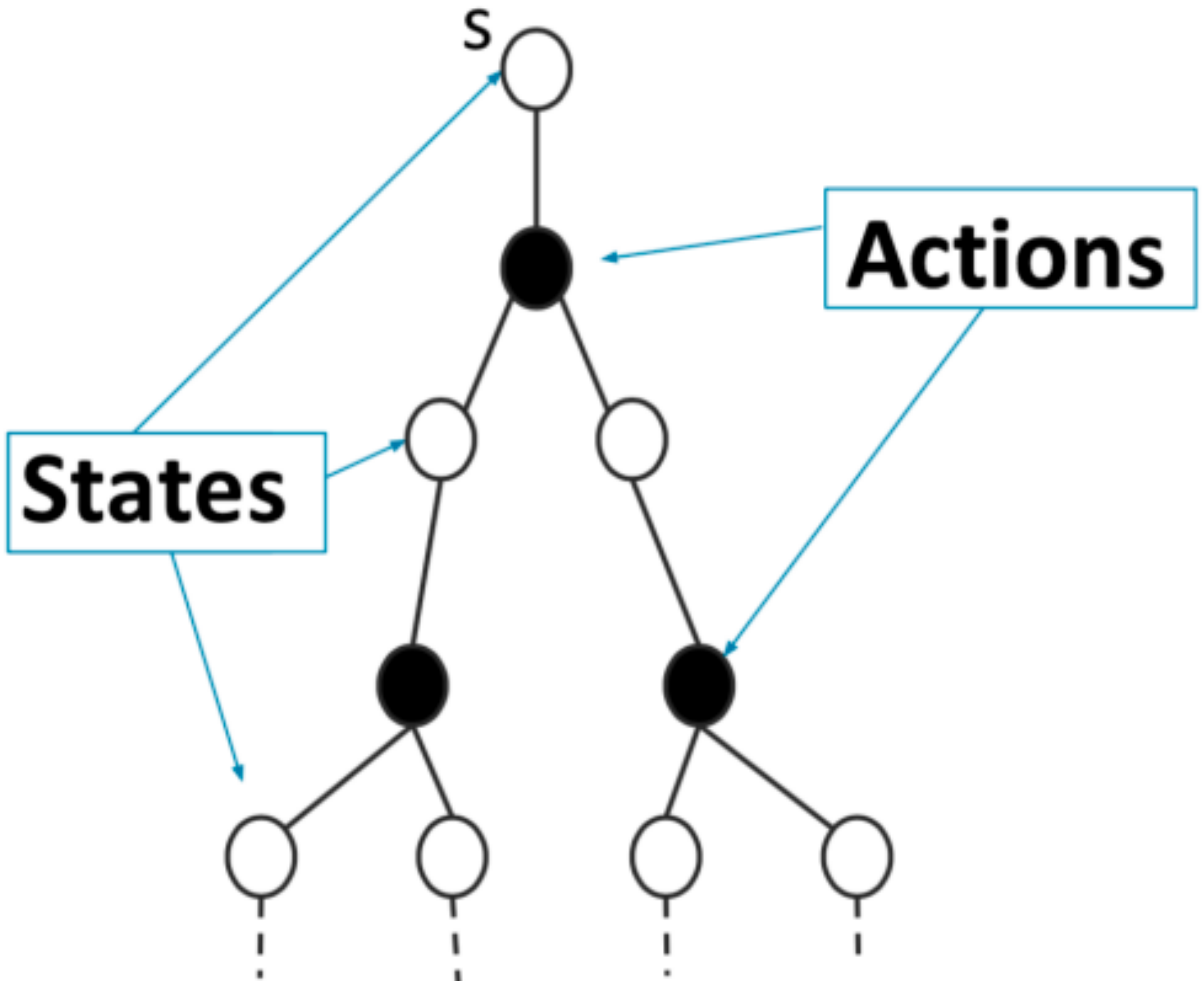
Policy Evaluation Diagram

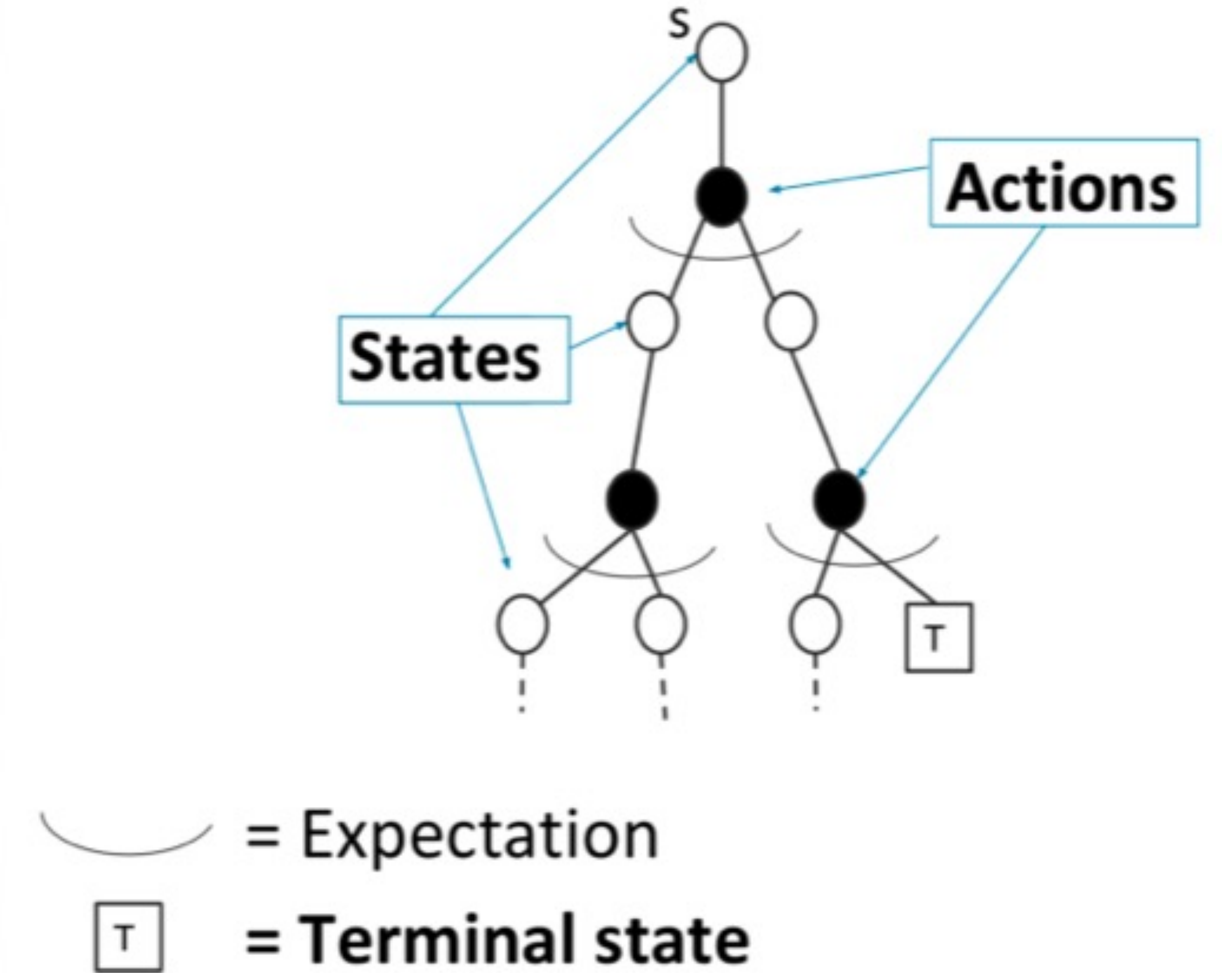


Policy Evaluation Diagram



Policy Evaluation Diagram

$$V^{\pi}(s) = V^{\pi}(s) + \alpha(G_{i,t} - V^{\pi}(s))$$



Policy Evaluation Diagram

$$V^{\pi}(s) = V^{\pi}(s) + \alpha(G_{i,t} - V^{\pi}(s))$$